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POSITION FINDING WITH EMPIRICAL PRIOR KNOWLEDGE

Peter J. Butterly

Research Contribution 197

Center for Naval Analyses

Systems Evaluation Group

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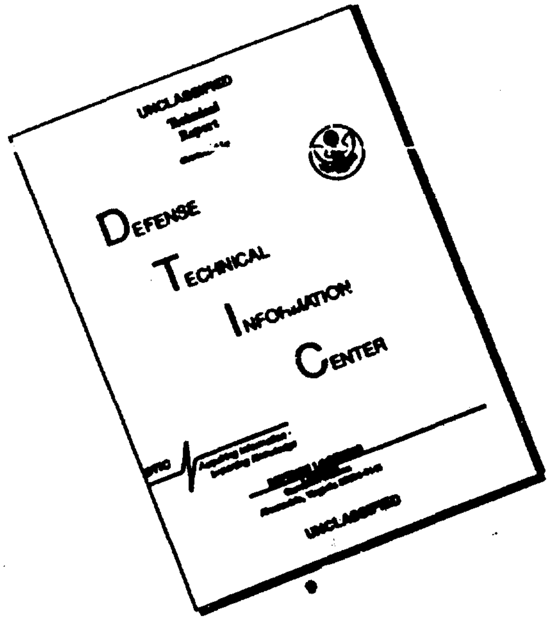

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Systems Evaluation Group

Research Contribution 197

POSITION FINDING WITH EMPIRICAL PRIOR KNOWLEDGE

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Peter J. Butterly

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I. INTRODUCTION

The position of a source of radiation is often estimated from bearing observations made from two or more known locations. If we let (x, y) denote the coordinates of the source and α_i its true bearing from a station with site location (X_i, Y_i) for i = 1, 2, ...n, then, if precise measurements were possible, the geometrical relationships linking the known quantities X_i, Y_i, α_i and the unknown quantities x and y could be used to determine the required coordinates of the source. The geometrical relationships can be formulated for each station separately and therefore have the form

$$\alpha_i = f(X_i, Y_i, x, y) = f_i(x, y) \qquad (1.1)$$

$$\text{for } i = 1, 2, \ldots n \, .$$

In practice, the angular observations will be subject to errors of several kinds. Since, however, the treatment of systematic errors in the measurement equipment and errors due to spurious sources of radiation usually consists of procedures for refining the observed data before estimating the fix, we shall take as our starting point the possibly-modified data set:

$$\theta_i = \alpha_i + \Psi_i \qquad (1.2)$$

$$\text{for } i = 1, 2 \ldots n,$$

or, in vector notation

$$\boldsymbol{\theta} = \boldsymbol{\alpha} + \boldsymbol{\Psi} \qquad (1.2a)$$

where the Ψ_i represent random measurement errors having the joint probability density function:

$$p(\boldsymbol{\Psi}) = p(\Psi_1, \Psi_2, \ldots. \Psi_n) \; . \qquad (1.3)$$

The problem of fix estimation is one of constructing an algorithm which, for the appropriate site geometry and for an assumed error distribution, provides an optimal estimate of the source position from the data furnished by the observations.

II. SOME CRITICISMS OF CURRENT ESTIMATION PROCEDURES

If it is accepted that the true bearings have fixed, but unknown values, then, since all of the pertinent geometrical information is contained in the nonlinear relations (1.1), the problem is seen to consist of estimating a known nonlinear function of the parameter $\boldsymbol{\alpha}$ of the joint density function $p(\boldsymbol{\theta}; \boldsymbol{\alpha})$ the form of which is readily obtained from that of the assumed error density $p(\boldsymbol{\psi})$. Consequently, it is not surprising that for many systems in current use, the estimating procedures are based on the principle of maximum likelihood. References (a) through (d) give procedures obtained by this approach.

The use of maximum likelihood methods for fix estimation has been subject to some criticisms. These include the following:

(a) Several factors limit the numbers of bearing observations. There is the availability of suitable station sites and the cost of measurement equipment. Also, if surveillance is exercised over an extended area or if the duration of the radiation is small enough, only some stations may have observations to report and of these, some may be rejected as being invalid. For what is more often than not a small sample problem, the properties of consistency and asymptotic efficiency are clearly irrelevant. [e]

(b) Although it is fairly straightforward to formulate the likelihood equations, the nonlinear character of the geometrical relationships (1.1) generally precludes a solution in closed form [f, c]. This leads to solutions in specific cases which are direct approximations or are based on iterative techniques or which combine such procedures. Unfortunately, the derivations of these solutions seldom include adequate treatments of the implications of the approximations or the convergence properties of a proposed iterative procedure.

(c) The method of maximum likelihood makes no provision for making use of prior knowledge. This is almost always present in some degree. Thus, not only is valid information not incorporated, in addition there is the possibility that a correct estimate of the most probable position of the source may be in conflict with this prior knowledge. The example of the estimated position being on land for an emitter known to be shipborne is not necessarily a consequence of anomalies in either the data or the fix algorithm.

III. POSITION FINDING USING BAYES' PROCEDURES

If, in contrast to the assumptions of the previous section, the coordinates of the radiating source are assumed to be random variables for which a realistic joint density can be assigned from knowledge existing before the observations are made, the problem may be formulated within a Bayesian framework. This formulation permits the available prior information to be incorporated in the estimation procedure. For a postulated prior density $\sigma(x, y)$, the required Bayes' estimator is given by the values of $\hat{x}$ and $\hat{y}$ which, for each possible set of bearing observations $\boldsymbol{\theta}$, minimize the a posteriori risk, namely:

$$\iint L(\hat{x}, \hat{y}; x, y)\, p(x, y \mid \boldsymbol{\theta})\, dx\, dy \tag{3.1}$$

where,

$$p(x, y \mid \boldsymbol{\theta}) = \frac{p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)}{p(\boldsymbol{\theta})} \tag{3.2}$$

$$= \frac{p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)}{\iint p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)\, dx\, dy} \tag{3.2a}$$

is the <u>a posteriori</u> density, expressed here in terms of known quantities, and $L(\hat{x}, \hat{y}, ; x, y)$ is an appropriate cost function. Where this is a quadratic function of the estimation errors $(\hat{x} - x)$ and $(\hat{y} - y)$, or alternatively, certain conditions listed in reference (g) are known to hold, the Bayes' estimator is simply the mean of this a posteriori density function. It may be obtained from quantities assumed to be known by means of the two expressions:

$$\hat{x} = \iint x\, p(x, y \mid \boldsymbol{\theta})\, dx\, dy = \frac{\iint x\, p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)\, dx\, dy}{\iint p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)\, dx\, dy} \tag{3.3a}$$

$$\hat{y} = \iint y\, p(x, y \mid \boldsymbol{\theta})\, dx\, dy = \frac{\iint y\, p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)\, dx\, dy}{\iint p(\boldsymbol{\theta} \mid x, y)\, \sigma(x, y)\, dx\, dy} \tag{3.3b}$$

These standard forms represent but the starting point for the derivation of a practical algorithm for fix estimation. Although they make formal provision for the inclusion of prior knowledge this will, in general, be of an empirical character. Consequently, any algorithm based on the above Bayesian formulation presupposes that the conversion of this empirical knowledge to an acceptable prior density has been achieved. Furthermore, there is no evidence that this approach offers any advantage over that based on maximum likelihood in providing solutions in closed form. To illustrate these points, let us examine the special case most frequently considered in the material referenced. For this case, the assumptions are as follows:

(a) The surveillance area is small enough for the curvature of the earth's surface to be neglected.

(b) For all stations the observation errors do not depend on their respective distances from the source of radiation and are normally (though not necessarily identically) distributed with mean zero.

(c) The observation errors for each station are independent.

The first assumption determines the geometrical relationship applicable to this case. Assuming angular measurements from the direction indicated by a positive y coordinate, it is readily shown that, for this case, the relation (1.1) has the form:

$$\alpha_i = f_i\,(x,\ y) = \arctan \frac{x - X_i}{y - Y_i} \quad . \tag{3.4}$$

for i = 1, 2 ... n, where n is the number of valid observations. The remaining assumptions, together with equations (1.3) and (1.2a) yield the following:

$$p(\Psi_i) = (2\pi)^{-1/2} \sigma_i^{-1} e^{-\Psi_i^2/2\sigma_i^2} \tag{3.5}$$

$$p(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \prod_{i=1}^{n} \left[(2\pi)^{-1/2} \sigma_i^{-1} e^{-\frac{1}{2\sigma_i^2}(\theta_i - \alpha_i)^2} \right] \tag{3.6}$$

$$= (2\pi)^{-n/2} (\sigma_1 \ldots \sigma_n)^{-1} e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} (\theta_i - \alpha_i)^2} \tag{3.6a}$$

$$p(\boldsymbol{\theta}|\,x,\ y) = (2\pi)^{-n/2} (\sigma_i \ldots \sigma_n)^{-1} e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - f_i\,(x, y)\right\}^2} \tag{3.7}$$

where, from (3.4).

$$f_i(x,\ y) = \arctan \frac{x - X_i}{y - Y_i} \quad .$$

Substitution in (3.2a) gives the a posteriori density corresponding to the arbitrary prior density $\sigma(x,\ y)$, namely:

$$p(x, y|\theta) = \frac{e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - \arctan \frac{x - X_i}{y - Y_i}\right\}^2} \sigma(x, y)}{\iint e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - \arctan \frac{x - X_i}{y - Y_i}\right\}^2} \sigma(x, y)\, dx,\, dy} \tag{3.8}$$

and substituting in (3.3a) and (3.3b) gives the expressions for the corresponding fix estimator, namely:

$$\hat{x} = \frac{\iint x\, e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - \arctan \frac{x - X_i}{y - Y_i}\right\}^2} \sigma(x, y)\, dx\, dy}{\iint e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - \arctan \frac{x - X_i}{y - Y_i}\right\}^2} \sigma(x, y)\, dx\, dy} \tag{3.9}$$

$$\hat{y} = \frac{\iint y\, e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - \arctan \frac{x - X_i}{y - Y_i}\right\}^2} \sigma(x, y)\, dx\, dy}{\iint e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{\theta_i - \arctan \frac{x - X_i}{y - Y_i}\right\}^2} \sigma(x, y)\, dx\, dy} \tag{3.9a}$$

These expressions serve to illustrate the general problem. First, a realistic prior density has to be assigned on the basis of existing prior knowledge, then, means must be

found for reducing the resulting complex mathematical forms to feasible computational procedures. In the following section a practical solution to these problems will be advanced. Essentially it consists of formulating the prior knowledge in a way which simplifies the mathematical relationships which result from a Bayesian approach.

IV. IMPLEMENTATION OF THE BAYESIAN FORMULATION

The Bayesian formulation of the position finding problem may be summarized as follows. The relationships which characterize the site geometry $f_i(x, y)$ and the joint density of the observation errors, $p(\psi)$, are assumed known. Prior information is incorporated by means of an assigned density $\sigma(x, y)$. The principal requirement is a feasible computational procedure which makes use of these known or assigned quantities to provide an estimate of the position of the source of radiation corresponding to the particular data vector yielded by the observations. This estimate is to be chosen so that the a posteriori risk is minimized.

Bayes' rule provides a formal statement of how prior knowledge is modified by observations so any algorithm which provides the a posteriori density $p(x, y|\theta)$ from a knowledge of the prior density $\sigma(x, y)$ and the data vector θ is formally equivalent to it. That is, the algorithm must be based on the relationships (3.2) or (3.2a), namely:

$$p(x, y|\theta) = \frac{p(\theta|x, y)\,\sigma(x, y)}{p(\theta)}$$

$$= \frac{p(\theta|x, y)\,\sigma(x, y)}{\iint p(\theta|x, y)\,\sigma(x, y)\,dx\,dy}$$

We note that the conditional density $p(\theta|x, y)$ is readily obtained from the error density $p(\psi)$ which is assumed to be known. In addition, it is clear that if the density function $p(\theta)$ can be expressed in closed form, an explicit algorithm for effecting the mapping of $\sigma(x, y)$ into $p(x, y|\theta)$ is assured. The derivation of an explicit algorithm is therefore seen to depend on obtaining a closed form expression for

$$\iint p(\theta|x, y)\,\sigma(x, y)\,dx\,dy$$

where $p(\theta|x, y)$, which is dependent on the site geometry, has a complex form such as that given for a specific example by equation (3.7).

Rather than pursue this course for the various error distributions and site geometries that could exist and for a completely arbitrary choice of prior density we adopt an approach based on the following argument. Prior knowledge of the source position is necessarily imprecise. Although it may be based on measurements which have substantially greater inherent accuracy than those under discussion, the assumption that a probability can be assigned for all possible values of the coordinates is in conflict with the limited resolution that can be achieved in practice. Consequently, it is difficult to envision a situation where all of the prior knowledge cannot be accounted for by assigning probabilities to only a finite set of source coordinates. This leads to the specification of a prior density having the form:

$$\sigma(x, y) = \sigma_{jk}\,\delta(x - x_j, y - y_k), \qquad (4.1)$$

where j and k are chosen from index sets of finite extent such as:

$$j = -J, -J+1, \ldots\ldots -1, 0, 1, \ldots\ldots J-1, J$$

$$k = -K, -K+1, \ldots\ldots -1, 0, 1, \ldots\ldots K-1, K$$

and $\delta(x - x_j, y - y_k)$ denotes a generalized bivariate function, usually referred to as a delta function, whose definition and properties are given in reference (h). Making use of these properties we have,

$$p(\theta) = \iint p(\theta | x, y)\, \sigma(x, y)\, dx\, dy$$

$$= \sum_j \sum_k p(\theta | x_j, y_k)\, \sigma_{jk} \tag{4.2}$$

which is in closed form. The a posteriori density is then given by (3.2), namely:

$$p(x, y|\theta) = \frac{p(\theta|x, y)\, \sigma(x, y)}{p(\theta)} \tag{4.3}$$

$$= \frac{p(\theta|x_j, y_k)\, \sigma_{jk}\, \delta(x - y_j, y - y_k)}{\sum_j \sum_k p(\theta| x_j, y_k)\, \sigma_{jk}},$$

for any point (x_j, y_k). If, for particular data vector θ which results from the observations, we write:

$$p_{jk} = p(\theta|x_j, y_k) \tag{4.4}$$

and define q_{jk} by the relation:

$$q_{jk} = \frac{p_{jk}\, \sigma_{jk}}{\sum_j \sum_k p_{jk}\, \sigma_{jk}} \tag{4.5}$$

then, on substituting in (4.3) we obtain

$$p(x, y|\theta) = q_{jk}\, \delta(x - x_j, y - y_k), \tag{4.6}$$

for each value of j and k, as the a posteriori density corresponding to the observed data. Since this has the identical form to that given by (4.1) for the prior density, the required algorithm for mapping the postulated prior density into its a posteriori counterpart is therefore given by equation (4.5) above, with values of p_{jk} as determined by equation (4.4) from a knowledge of the error density, site geometry, and observed data. The values of p_{jk} thus reflect the specific system under consideration as well as the data actually

recorded, whereas equation (4.5) is a restatement of Bayes' Rule applicable to the form of prior density chosen to describe the prior knowledge.

These results follow from the assumption that all of the prior knowledge may be accounted for by assigning probabilities to a finite set of source coordinates and do not depend on how this set is chosen. While in some cases a rationale for selecting this set may be based on some aspect of the prior knowledge, it will often suffice to allocate one point for each rectangular area enclosed by a grid system spaced in accordance with a degree of resolution that is considered adequate for a treatment of the entire problem. This grid system assumes a comparable role in the formulation of the a posteriori density, giving this in a form which provides a convenient starting point for further bearing observations or for implementing discrete search procedures such as those described in references (i) through (k).

Although there will be cases where the overall objective is merely to record the estimated source position, in general, some subsequent action, such as search, is envisioned. This action should be based on the a posteriori density -- otherwise all of the available information is not utilized. Of course, once this density is known, the estimated source position and certain related information is readily ascertained.

For a quadratic cost function, the estimated fix is obtained from:

$$\begin{aligned}\hat{x} &= E(x|\theta)\\ &= \iint x\, p(x, y|\theta)\, dx\, dy\\ &= \sum_j \sum_k x_j q_{jk}\end{aligned} \tag{4.7}$$

on substituting for $p(x, y|\theta)$ from equation (4.6), and from

$$\hat{y} = \sum_j \sum_k y_k q_{jk} . \tag{4.7a}$$

Some additional properties of the a posteriori density which may prove useful can be obtained from a knowledge of the moments [1] which are given by:

$$\begin{aligned}\mu_{rs} &= E(x^r y^s|\theta)\\ &= \iint x^r y^s p(x, y|\theta)\, dx\, dy\\ &= \sum_j \sum_k x_j^r y_k^s q_{jk} \quad ,\end{aligned} \tag{4.8}$$

and the probability that the source is located in any given region may be determined by summing the values of q_{jk} for points within the region.

To apply these results to the example given in Section III we require only the values p_{jk} appropriate to the error distribution and site geometry which characterize this example. These are readily obtained from equation (3.7), giving

$$p_{jk} = p(\boldsymbol{\theta} | x_j, y_k)$$

$$= (2\pi)^{-n/2} (\sigma_1 \ldots \sigma_n)^{-1} e^{-1/2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left\{ \theta_i - f_i (x_j, y_k) \right\}^2}$$

where

$$f_i (x_j, y_k) = \arctan \frac{x_j - X_i}{y_k - Y_i} ,$$

and the observed values of $\theta_1, \theta_2 \ldots\ldots \theta_n$ are inserted. The appropriate a posteriori density is then given by equation (4.5) and the estimated position of the source by equations (4.7) and (4.7a).

We observe that in cases such as this where the observations from each station are independent, it is not necessary to await the reporting of all n bearings before commencing these procedures. A simple sequential treatment may be adopted. Assume the r^{th} station to be the first to report. For this single observation the appropriate values of p_{jk} are given by the univariate conditional density $p(\theta_r | x, y)$, $1 \le r \le n$. Inserting these values in equation (4.5) provides the a posteriori density resulting from this observation. When the second observation is reported, this density becomes the new prior density from which the a posteriori density corresponding to the two observations is computed using the same procedure. This process can be repeated until all stations have reported or until sufficient information has been accumulated. An estimate of the position of the source is readily obtained at any stage by means of equations (4.7) and (4.7a).

We conclude with an example of the computation for the special case referred to in Section III which was characterized by a "flat-earth" geometry, normally distributed errors, and independent observations from each station. The locations of the stations, the standard deviations of the error distributions and the recorded observations are taken to be:

i	X_i (miles)	Y_i (miles)	σ_i (degrees)	θ_i (degrees)
1	-10	-80	2	5.13
2	-40	-60	3	36.7
3	-80	-10	5	77.9

It is assumed that the source of the radiation is shipborne and located within an 11 x 11 square mile area centered about the origin of the coordinate system and that this area contains a small, irregularly shaped land mass. In addition, the position of the source is assumed to be uniformly distributed over the residual sea area, so the values of the prior density function at points spaced at one mile intervals are as shown in table I from which the location and shape of the land mass is readily apparent.

Since the procedure for computing the required a posteriori density is repeated for each point of the 11 x 11 array resulting from this choice of spacing, we consider an arbitrary point (x_j, y_k) at which the value of the prior density is $\sigma_{jk}\ \delta(x - x_j, y - y_k)$. For this point, the true values of the bearings from each station are given by equation (3.4), namely

$$\alpha_i = f_i\ (x_j, y_k) = \arctan \frac{x_j - X_i}{y_k - Y_i}$$

for i = 1, 2, 3. The numerical value of

$$p_{jk} = p(\theta \mid x_j, y_k)$$

is then obtained from equation (3.7). This is repeated for each point to give the array shown in table II. From this, the required a posteriori density is computed by repeated application of equation (4.5) and the estimated source position by means of equations (4.7) and (4.7a). The numerical values so obtained are given in table III.

This example is reconsidered for the case in which the three observations are not simultaneously available by means of the sequential procedure outlined above. Table IV gives the values of the density function prior to any station reporting and table V the a posteriori density based on the single observation reported by Station 1. This density assumes the role of a new prior density function for the computation of the a posteriori density corresponding to the second observation to be reported. Table VI gives the numerical values at this stage and a similar computation for the observation reported by Station 3 yields the final a posteriori density corresponding to all of the observations shown in table VII. As would be expected, this is identical to table III. The estimated position of the source at each stage is readily computed from the corresponding a posteriori density.

TABLE I

PRIOR DENSITY FUNCTION

0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0	0	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0	0	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893

TABLE II

VALUES OF p_{jk}

2.31E-5	6.53E-5	1.28E-4	2.11E-4	2.95E-4	3.51E-4	3.55E-4	3.07E-4	2.27E-4	1.44E-4	7.92E-5
3.83E-5	8.74E-5	1.65E-4	2.63E-4	3.53E-4	4.01E-4	3.87E-4	3.19E-4	2.24E-4	1.35E-4	7.00E-5
5.21E-5	1.13E-4	2.07E-4	3.16E-4	4.07E-4	4.43E-4	4.08E-4	3.19E-4	2.12E-4	1.21E-4	5.92E-5
6.78E-5	1.43E-4	2.51E-4	3.68E-4	4.54E-4	4.71E-4	4.13E-4	3.06E-4	1.93E-4	1.04E-4	4.80E-5
8.53E-5	1.73E-4	2.93E-4	4.13E-4	4.87E-4	4.82E-4	4.02E-4	2.83E-4	1.69E-4	8.56E-5	3.71E-5
1.04E-4	2.04E-4	3.31E-4	4.47E-4	5.04E-4	4.74E-4	3.75E-4	2.50E-4	1.41E-4	6.71E-5	2.73E-5
1.22E-4	2.31E-4	3.60E-4	4.66E-4	5.00E-4	4.48E-4	3.35E-4	2.11E-4	1.12E-4	5.02E-5	1.92E-5
1.38E-4	2.52E-4	3.77E-4	4.66E-4	4.76E-4	4.04E-4	2.87E-4	1.70E-4	8.48E-5	3.57E-5	1.27E-5
1.51E-4	2.65E-4	3.79E-4	4.47E-4	4.24E-4	3.49E-4	2.34E-4	1.31E-4	6.11E-5	2.41E-5	8.02E-6
1.58E-4	2.67E-4	3.65E-4	4.10E-4	3.78E-4	2.88E-4	1.82E-4	9.53E-5	4.18E-5	1.54E-5	4.78E-6
1.60E-4	2.58E-4	3.37E-4	3.60E-4	3.14E-4	2.26E-4	1.34E-4	6.60E-5	2.71E-5	9.29E-6	2.88E-6

TABLE III

A POSTERIORI DENSITY FUNCTION

0.00105	0.00245	0.00479	0.00791	0.0111	0.0131	0.0133	0.0115	0.00851	0.00541	0.00297
0.00146	0.00328	0.00613	0.00984	0.0132	0.015	0.0145	0.0119	0.00839	0.00506	0.00262
0.00195	0.00425	0.00775	0.0118	0.0153	0.0166	0.0153	0.0119	0.00796	0.00454	0.00222
0.00254	0.00534	0.00938	0.0138	0.017	0.0176	0.0155	0.0115	0.00724	0.0039	0.0018
0.0032	0.0065	0.011	0.0155	0.0183	0.0181	0.015	0.0106	0.00631	0.0032	0.00139
0.00389	0.00763	0.0124	0.0168	0.0189	0.0178	0.014	0.00935	0.00526	0.00251	0.00102
0.00457	0.00865	0.0135	0.0174	0.0197	0.0168	0.0126	0	0	0.00188	0.000717
0.00513	0.00944	0.0141	0.0174	0.0178	0.0151	0	0	0	0.00134	0.000477
0.00566	0.00991	0.0142	0.0167	0.0163	0.0131	0	0	0	0.000902	0.000301
0.00594	0.00993	0.0137	0.0154	0.0142	0.0108	0.0068	0	0.00157	0.000576	0.000179
0.00599	0.00965	0.0126	0.0135	0.0118	0.00846	0.00502	0.00247	0.00101	0.000348	0.0001

$\hat{x} = -0.78$

$\hat{y} = 0.24$

TABLE IV

PRIOR DENSITY FUNCTION

0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0	0	0.00893	0.00893
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0	0	0.00893	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0	0.00893	[illegible]	[illegible]
0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	0.00893	[illegible]

TABLE V

A POSTERIORI DENSITY FUNCTION (1)

0.0108	0.0137	0.0156	0.0158	0.0143	0.0117	0.00848	0.00554	0.00325	0.00171	[illegible]
0.011	0.0139	0.0157	0.0157	0.0141	0.0113	0.00807	0.00517	0.00290	0.00152	0.000764
0.0112	0.0141	0.0157	0.0156	0.0138	0.0109	0.00766	[illegible]	0.00263	0.00135	[illegible]
0.0114	0.0142	0.0158	0.0155	0.0136	0.0105	0.00724	0.00444	[illegible]	0.00118	[illegible]
0.0116	0.0144	0.0158	0.0154	0.0133	0.0101	0.00682	0.00403	[illegible]	0.00103	[illegible]
0.0118	0.0146	0.0159	0.0153	0.0129	0.00968	0.0064	0.00375	0.00195	[illegible]	[illegible]
0.012	0.0147	0.0159	0.0151	0.0126	0.00925	0.00599	0	0	0.000773	[illegible]
0.0122	0.0149	0.0159	0.0149	0.0122	0.00881	0	0	0	0.000661	[illegible]
0.0124	0.015	0.0159	0.0147	0.0119	0.00836	0	0	0	0.00056	[illegible]
0.0126	0.0152	0.0159	0.0145	0.0115	0.00791	0.00477	0	0.00116	0.000471	0.000168
0.0128	0.0153	0.0158	0.0142	0.011	0.00745	0.00438	0.00224	0.001	0.000392	0.000134

$\hat{x} = -2.1$

$\hat{y} = -0.25$

TABLE VI

A POSTERIORI DENSITY FUNCTION (2)

0.000566	0.00132	0.0026	0.00432	0.00609	0.00728	0.00743	0.00647	0.00483	0.0031	0.00172
0.000809	0.00184	0.0035	0.00562	0.00762	0.00876	0.00855	0.00711	0.00506	0.00308	0.00162
0.00115	0.00253	0.00466	0.00722	0.00942	0.0104	0.00967	0.00766	0.00518	0.00299	0.00148
0.00161	0.00345	0.00614	0.00916	0.0115	0.0121	0.0107	0.00809	0.00518	0.00283	0.00133
0.00225	0.00465	0.00798	0.0114	0.0137	0.0138	0.0117	0.00835	0.00507	0.00261	0.00115
0.0031	0.00619	0.0102	0.0141	0.0161	0.0155	0.0124	0.00842	0.00482	0.00234	0.00097
0.00422	0.00812	0.0129	0.017	0.0186	0.017	0.0129	0	0	0.00204	0.000792
0.00567	0.0105	0.016	0.0202	0.021	0.0182	0	0	0	0.00172	0.000625
0.00751	0.0134	0.0195	0.0235	0.0232	0.019	0	0	0	0.0014	0.000475
0.00379	0.0167	0.0234	0.0267	0.025	0.0194	0.0124	0	0.00295	0.0011	0.000349
0.0125	0.0205	0.0273	0.0296	0.0263	0.0192	0.0110	0.00577	0.0024	0.000836	0.000245

$\hat{x} = -1.1$

$\hat{y} = -1.1$

TABLE VII

A POSTERIORI DENSITY FUNCTION (3)

0.00105	0.00245	0.00479	0.00791	0.0111	0.0131	0.0133	0.0115	0.00851	0.00541	[illegible]
0.00146	0.00328	0.00619	0.00984	0.0132	0.015	0.0145	0.0119	0.00839	0.00505	[illegible]
0.00195	0.00425	0.00775	0.0118	0.0153	0.0166	0.0153	0.0119	0.00796	[illegible]	[illegible]
0.00254	0.00534	0.00938	0.0138	0.017	0.0176	0.0155	0.0115	0.00724	[illegible]	[illegible]
0.0032	0.0065	0.011	0.0155	0.0183	0.0181	0.015	0.0106	0.00631	[illegible]	[illegible]
0.00389	0.00763	0.0124	0.0168	0.0189	0.0178	0.014	0.00935	0.00526	[illegible]	[illegible]
0.00457	0.00865	0.0135	0.0174	0.0187	0.0168	0.0126	0	0	0.00186	[illegible]
0.00519	0.00944	0.0141	0.0174	0.0178	0.0151	0	0	0	0.00134	[illegible]
0.00566	0.00991	0.0142	0.0167	0.0163	0.0131	0	0	0	[illegible]	[illegible]
0.00594	0.00999	0.0137	0.0154	0.0142	0.0108	0.0068	0	0.00157	0.000576	[illegible]
0.00599	0.00965	0.0126	0.0135	0.0118	0.00846	0.00502	0.00247	0.00101	[illegible]	[illegible]

$\hat{x} = -0.78$

$\hat{y} = 0.24$

(REVERSE BLANK)

V. SUMMARY AND CONCLUSIONS

In this communication the position finding problem is considered from a standpoint which includes the following features:

(a) The position of the source of radiation is assumed to act as a random variable for which a prior density can be assigned.

(b) The position finding system is assumed to consist of a set of geometrical relationships from which, in the absence of uncertainty, the position of the source could be determined, together with an error distribution to account for the uncertainty.

(c) Emphasis is placed on obtaining closed form expressions for the decision procedures established as a means of avoiding the uncertain accuracy of approximate procedures and the computational complexity of iterative procedures.

The problem is recognized to be fundamentally one of ascertaining the manner in which prior knowledge is modified by observed data and the proposed solution is based on a Bayesian treatment which assumes a particular formulation of the prior knowledge. The formulation is well suited to the frequently occurring case where this knowledge is empirical in nature and is nonrestrictive in the sense that any prior density can be approximated with arbitrary precision. This approach yields a simple algorithm for providing the required a posteriori knowledge in a similar formulation. This, in turn, characterizes the uncertainty in the presence of which any subsequent action is initiated in a form that is particularly convenient where this action consists of making further bearing observations or implementing certain discrete search procedures. The required estimator of the source position is also readily ascertained by means of this approach.

When compared with procedures in current use the Bayesian treatment proposed here offers the following advantages:

(a) It provides for the incorporation of prior knowledge.

(b) It provides, in addition to an estimate of the source position, all relevant a posteriori knowledge.

(c) It provides simple computational procedures for obtaining these results.

(d) It is applicable to a diverse range of position finding systems.

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13. ABSTRACT

Information on the position of a source of radiation is often obtained from bearing observations made from two or more known locations. ~~In this communication~~ some criticisms of existing procedures are enumerated and a new approach proposed. This approach allows for the incorporation of knowledge existing before the observations are made and provides simple computational procedures for estimating the source position and for ascertaining the new state of knowledge from the data furnished by the observations.

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